5.1 Orthogonal Sets

We already saw that

$$S=\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\2\end{bmatrix}, \begin{bmatrix}0\\5\\0\end{bmatrix}\right\}$$

is such that

$$\vec{v}_1 \cdot \vec{v}_2 = 0, \quad \vec{v}_1 \cdot \vec{v}_3 = 0, \text{ and } \vec{v}_2 \cdot \vec{v}_3 = 0$$

Thus, the set S is called an orthogonal set.

Def. The set $S=\{\vec{v}_1, .., \vec{v}_p\} \subset \mathbb{R}^n$ is $\boxed{\text{an orthogonal set}}$

if $\boxed{v_i \cdot v_j = 0, \quad i \neq j, \quad i,j=1.., p.}$

Thm 5.1 $S=\{\vec{v}_1, .., \vec{v}_p\} \subset \mathbb{R}^n$, $\quad \vec{v}_i \neq \vec{0}$, $i=1,.,p$

i) If S is $\boxed{\text{an orthogonal set, then } S \text{ is linearly independent.}}$ and S is a basis for Span(S).

Proof. Want to prove:

If $c_1\vec{v}_1 + \dots + c_p\vec{v}_p = \vec{0} \Rightarrow c_1 = 0, .., c_p = 0.$

Answ: If $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p = \vec{0}$

Then

$$0 = \vec{v}_1 \cdot \vec{0} = \vec{v}_1 \cdot (c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p) = v_1 \cdot (c_1\vec{v}_1) + v_1 \cdot (c_2\vec{v}_2) + \dots + \vec{v}_1 \cdot (c_p\vec{v}_p)$$

$$\overset{\text{Thm 1}}{=} c_1(\vec{v}_1 \cdot \vec{v}_1) + c_2(\vec{v}_1 \cdot \vec{v}_2) + \dots + c_p(\vec{v}_1 \cdot \vec{v}_p) \overset{\text{Hyp orthog.}}{=} c_1(\vec{v}_1 \cdot \vec{v}_1)$$

$$\Rightarrow \quad \boxed{c_1(\vec{v}_1 \cdot \vec{v}_1) = 0 \Rightarrow c_1 = 0,} \text{ since } \vec{v}_1 \neq \vec{0}.$$

A Similar procedure, using $\vec{v}_2, \vec{v}_3, \ldots, \vec{v}_{p-1}$ Succesively leads to

$$0 = \vec{v}_j \cdot \vec{0} = v_j \cdot (c_1 \vec{v}_1 + \cdots + c_j \vec{v}_j + \cdots + c_p \vec{v}_p) =$$

$$= c_1 (v_j \cdot \vec{v}_1) + \cdots + c_j (\vec{v}_j \cdot \vec{v}_j) + \cdots + c_p (\vec{v}_j \cdot \vec{v}_p) \overset{\text{Hyp. orthog.}}{=\!=}$$

$$= c_j (\vec{v}_j \cdot \vec{v}_j) \Rightarrow \boxed{c_j (\vec{v}_j \cdot \vec{v}_j) = 0 \Rightarrow c_j = 0, \text{ since } \vec{v}_j \neq 0 \quad j = 2, \ldots, p-1.}$$

For $j = p$, it is completely analogous.

Since $c_1 = 0, \ldots, c_p = 0$, then S is linearly independent.

Also, S spans Span(S) then, S is a basis for Span(S).

Def. $B = \{\vec{v}_1, \ldots, \vec{v}_r\} \subset \mathbb{R}^n$ is formed by orthogonal vectors (nonzero) and Span$(B) = W$ then B is called an orthogonal basis for the Subspace W.

Our Example: Since

$$S = \left\{ \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, & \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, & \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \end{array} \right\}$$

is an orthogonal set of nonzero vectors then S is linearly independent (using thm 4)

Also, S is a basis for $\mathbb{R}^3$ because S consists of 3 vectors and $\dim(\mathbb{R}^3) = 3$.

Advantage of an orthogonal basis.

Find the coordinates of $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \right\}$$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad (3.1)$$

We want to find the coefficients c_1, c_2, c_3. They will be the coordinates of $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to the basis B. It means

$$[\vec{u}]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

The equation (3.1) is equivalent to solving the linear system

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

row-reducing augmented matrix

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 5 & 2 \\ -1 & 2 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 2 & 0 & 3 \\ 0 & 0 & 5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 5 & 2 \end{bmatrix}$$

$\Rightarrow \quad c_1 = -1, \quad c_2 = 1, \quad c_3 = 2/5.$

Thus,

$$[\vec{u}]_B = \underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}_B}^{\vec{u}} = \begin{bmatrix}-1\\1\\2/5\end{bmatrix}\begin{matrix}=c_1\\=c_2\\=c_3\end{matrix}$$

These coefficients can be obtained easily using the orthogonal property of the vectors of B.

In fact,

If $\underbrace{c_1\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_2\begin{bmatrix}2\\0\\2\end{bmatrix} + c_3\begin{bmatrix}0\\5\\0\end{bmatrix}} = \begin{bmatrix}1\\2\\3\end{bmatrix}$

then $\begin{bmatrix}1\\0\\-1\end{bmatrix}\cdot\left(\quad\downarrow\quad\right) = \begin{bmatrix}1\\0\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}$

$\overset{\text{thm 1}}{\Rightarrow}$ $c_1\left(\overbrace{[1\ 0\ -1]\begin{bmatrix}1\\0\\-1\end{bmatrix}}^{\vec{v}_1\cdot\vec{v}_1}\right) = \overbrace{[1\ 0\ -1]\begin{bmatrix}1\\2\\3\end{bmatrix}}^{\vec{v}_1\cdot\vec{u}}$

or $2c_1 = 1-3 = -2 \Rightarrow \boxed{c_1=-1}$ Same as before.

In Symbols,

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{u}$$

multiplying both sides by $\vec{v}_1$ (dot product).

$$\vec{v}_1\cdot(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = \vec{v}_1\cdot\vec{u}$$

Using orthogonality and thm 1

$$c_1(\vec{v}_1\cdot\vec{v}_1) = \vec{v}_1\cdot\vec{u}$$

$$\Rightarrow \boxed{c_1 = \frac{\vec{v}_1\cdot\vec{u}}{\vec{v}_1\cdot\vec{v}_1}}$$

Ask students to compute c_2, c_3.

Analogously,

$$C_2 = \frac{\vec{v}_2 \cdot \vec{u}}{\vec{v}_2 \cdot \vec{v}_2} = \frac{[2\,0\,2]\begin{bmatrix}1\\2\\3\end{bmatrix}}{[2\,0\,2]\begin{bmatrix}2\\0\\2\end{bmatrix}} = \frac{8}{8} = 1$$

and

$$C_3 = \frac{\vec{v}_3 \cdot \vec{u}}{\vec{v}_3 \cdot \vec{v}_3} = \frac{[0\,5\,0]\begin{bmatrix}1\\2\\3\end{bmatrix}}{[0\,5\,0]\begin{bmatrix}0\\5\\0\end{bmatrix}} = \frac{10}{25} = \frac{2}{5}$$

$$\boxed{[\vec{u}]_B = \begin{bmatrix}1\\2\\3\end{bmatrix}_B = \begin{bmatrix}C_1\\C_2\\C_3\end{bmatrix} = \begin{bmatrix}-1\\1\\2/5\end{bmatrix}}$$

Same as before, but computation is much easier.

Verification:

$$(-1)\begin{bmatrix}1\\0\\-1\end{bmatrix} + (1)\begin{bmatrix}2\\0\\2\end{bmatrix} + \frac{2}{5}\begin{bmatrix}0\\5\\0\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} \checkmark$$

This result can be generalized to $\mathbb{R}^n$

Thm 5.2 $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} \subset \mathbb{R}^n$ is an orthogonal basis for $W \subset \mathbb{R}^n$.

The coordinates of $\vec{u} \in W$ with respect to the basis B satisfy

$$\boxed{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p = \vec{u}}$$

where

$$\boxed{c_j = \frac{\vec{u} \cdot v_j}{v_j \cdot v_j}, \quad j = 1, \ldots, p.}$$

Proof: Completely analogous to the previous computation in $\mathbb{R}^3$.

The set of vectors $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \right\}$

can be normalized and transformed in a new set of unit vectors that are orthogonal.

$$\hat{B} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{8} \\ 0 \\ 2/\sqrt{8} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

This new set $\hat{B}$ is called an orthonormal set.

<u>Definition</u>. A set $S = \{ \vec{u}_1, .., \vec{u}_p \}$ in $\mathbb{R}^n$ is an orthonormal set if

i) S is an orthogonal set.

ii) Each vector u_i in S is a unit vector.

<u>Definition</u> An <u>orthonormal</u> <u>basis</u> for a subspace W of $\mathbb{R}^n$ is a basis of W that is an orthonormal set.

<u>Example</u>: $\hat{B}$ from our previous example is an orthonormal basis for $\mathbb{R}^3$.

<u>Theorem 5.3</u> i) $\hat{B} = \{\hat{q}_1, .., \hat{q}_p\}$ is an orthonormal basis for a subspace W of $\mathbb{R}^n$.

ii) $\vec{w}$ is any vector in W.

Then,
$$\vec{w} = (\vec{w}\cdot\hat{q}_1)\hat{q}_1 + \cdots + (\vec{w}\cdot\hat{q}_p)\hat{q}_p$$
and this representation is unique.

<u>Example</u> Find C_1, C_2, C_3 (coordinates) such that

$$C_1\begin{bmatrix}\sqrt{2}/2\\0\\-\sqrt{2}/2\end{bmatrix} + C_2\begin{bmatrix}\sqrt{2}/2\\0\\\sqrt{2}/2\end{bmatrix} + C_3\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

According to previous theorem,

$$C_1 = \begin{bmatrix}\sqrt{2}/2\\0\\-\sqrt{2}/2\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix} = \frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{2} = \frac{-2\sqrt{2}}{2} = -\sqrt{2}$$

$$C_2 = \begin{bmatrix}\sqrt{2}/2\\0\\\sqrt{2}/2\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix} = \frac{\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} = 2\sqrt{2}$$

$$C_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix} = 2$$

or
$$-\sqrt{2}\begin{bmatrix}\sqrt{2}/2\\0\\-\sqrt{2}/2\end{bmatrix} + 2\sqrt{2}\begin{bmatrix}\sqrt{2}/2\\0\\\sqrt{2}/2\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

Consider the matrix of exercise 17

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

This matrix has orthonormal columns.

Notice that

$$Q^T Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

<u>Definition</u> A matrix $Q_{n\times n}$ whose columns form an orthonormal set is called an orthogonal matrix.

<u>Thm 5.4</u> i) $U_{m\times n}$

the columns of U form an orthonormal set

if and only if $U^T U = I_n$

Theorem 5.4 $U_{m\times n}$

$$U \text{ has orthonormal columns } \overset{\text{if and only if}}{\Leftrightarrow} U^T U = I.$$

Proof.- Consider

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$U \text{ has orthonormal Columns } \Leftrightarrow \vec{u}_i \cdot \vec{u}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

Also,

$$U^T U = \begin{bmatrix} \vec{u}_1^T \rightarrow \\ \vec{u}_2^T \rightarrow \\ \vdots \\ \vec{u}_n^T \rightarrow \end{bmatrix} \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} =$$

$$= \begin{bmatrix} \vec{u}_1^T \vec{u}_1 & \vec{u}_1^T \vec{u}_2 & \cdots & \vec{u}_1^T \vec{u}_n \\ \vec{u}_2^T \vec{u}_1 & \vec{u}_2^T \vec{u}_2 & \cdots & \vec{u}_2^T \vec{u}_n \\ \vdots & & & \\ \vec{u}_n^T \vec{u}_1 & \cdots & \cdots & \vec{u}_n^T \vec{u}_n \end{bmatrix} = \begin{bmatrix} \vec{u}_1 \cdot \vec{u}_1 & \vec{u}_1 \cdot \vec{u}_2 & \cdots & \vec{u}_1 \cdot \vec{u}_n \\ \vec{u}_2 \cdot \vec{u}_1 & \vec{u}_2 \cdot \vec{u}_2 & \cdots & \vec{u}_2 \cdot \vec{u}_n \\ \vdots & & & \\ \vec{u}_n \cdot \vec{u}_1 & \vec{u}_n \cdot \vec{u}_2 & \cdots & \vec{u}_n \cdot \vec{u}_n \end{bmatrix}$$

Then

$$U^T U = I \Leftrightarrow \vec{u}_i \cdot \vec{u}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \Leftrightarrow U \text{ has orthonormal Columns.}$$

Thm 5.5

Therefore, $\boxed{U^T = U^{-1}, \quad \text{if}^{\text{ and only if}} \; U_{n\times n} \; (\text{square matrix})}$

Similar to
Theorem 5.6 1) $U_{m\times n}$ matrix with orthonormal columns.

2) $\vec{x}, \vec{y} \in \mathbb{R}^n$

Then

a) $\|U\vec{x}\| = \|\vec{x}\|$

b) $(U\vec{x})\cdot(U\vec{y}) = \vec{x}\cdot\vec{y}$

c) $(U\vec{x})\cdot(U\vec{y}) = 0 \iff \vec{x}\cdot\vec{y} = 0.$

Proof.

(b) $$U\vec{x}\cdot U\vec{y} = (U\vec{x})^T U\vec{y} = (\vec{x}^T U^T)(U\vec{y})$$
$$= \vec{x}^T(U^T U\vec{y}) = \vec{x}^T(I\vec{y}) = \vec{x}^T\vec{y} = \vec{x}\cdot\vec{y}$$

(a) $$\|U\vec{x}\|^2 = U\vec{x}\cdot U\vec{x} \overset{(b)}{=} \vec{x}\cdot\vec{x} = \|\vec{x}\|^2$$
$$\iff \|U\vec{x}\| = \|\vec{x}\|.$$

(c) $$(U\vec{x})\cdot(U\vec{y}) = 0 \overset{(b)}{\iff} \vec{x}\cdot\vec{y} = 0$$

Discuss thm 5.8

$Q_{n\times n}$ orthogonal matrix

a) Q^{-1} is orthogonal

b) $\det Q = \pm 1$

c) λ eigenv. of Q, then $|\lambda| = 1$

d) $Q_{1_{n\times n}}$ and $Q_{2_{n\times n}}$ orthogonal, then $Q_1 Q_2$ orthogonal.

Proof.

a) Q orthogonal $\Rightarrow Q^{-1} = Q^T$.

then, $(Q^{-1})^{-1} = (Q^T)^{-1} = (Q^{-1})^T \Rightarrow Q^{-1}$ is orthog.

b) $$1 = \det I = \det(QQ^{-1}) = \det(QQ^T)$$
$$= \det Q \det Q^T = (\det Q)^2$$
$$\Rightarrow \quad \det Q = \pm 1.$$

c) If $Q\vec{x} = \lambda\vec{x} \Rightarrow \|\vec{x}\| = \|Q\vec{x}\| = |\lambda|\,\|\vec{x}\| \Rightarrow |\lambda| = 1.$

d) $(Q_1 Q_2)^{-1} = Q_2^{-1} Q_1^{-1} = Q_2^T Q_1^T = (Q_1 Q_2)^T$

$\Rightarrow Q_1 Q_2$ is orthogonal.